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Dynamical Algebra of Invariants of Integro-differential Equations of Quantum Theory

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The Heisenberg–Weyl dynamical algebra of invariants for the special type of integro-differential equations of quantum theory is developed.

The present paper deals with the study of algebraic properties of a special type of integrodifferential equation, and in particular, with the construction of the dynamical algebra of invariants for this type of equation. However, the method of investigation may be easily generalized to a wider class of equations.

It is widely known that by performing an algebraic analysis of the multicomponent models of particle physics (Fusich, 1978), statistical mechanics (Rustamov, 1987), etc., one often reduces the problem to the algebraic analysis of integrodifferential equations. Moreover, in such cases the traditional methods of algebraic analysis of dynamical equations (Ovsiannikov, 1978; Ibragimav, 1983; Barut, 1977; Fushich, 1978; Man'co, 1979; Rustamov, 1983, 1987) are ineffective. Recall also that the dynamical algebra of invariants for the physical system under consideration provides a unified approach to such traditional problems of theoretical physics as the construction of complete systems of states and energy spectra, the Green's function in different representations, matrix elements and probabilities of different transitions, etc. (see above references).

Let us consider the equation

$$(i\partial_0 - \mathscr{H}^{(n/k)})\Psi(x, t) = 0 \tag{1}$$

where $\mathscr{H} = g^{\alpha\alpha}P_\alpha P_\alpha + m^2$; $\partial_0 = \partial/\partial t$; $P_\alpha = \partial/\partial x_\alpha$; $\alpha = 1, \ldots, l$; $x = (x_1, \ldots, x_l) \in \mathbb{R}^l$; $g^{\alpha\alpha}$ are diagonal elements of the metric tensor; n, k, and l are integers (n/k is noninteger); and m is a constant.

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In accordance with general statements, equation (1) should be understood in the following sense:

$$i\partial_0\Psi(x,t) - \int dr\, dy \exp\{ir^\alpha(x_\alpha - y_\alpha)\}\,\Psi(y,t)\bar{\mathscr{H}}(r) = 0 \tag{2}$$

where $\bar{\mathscr{H}}(r)$ denotes $\mathscr{H}^{(n/k)}$; $y=(y_1,\ldots,y_l)$ and $r=(r^1,\ldots,r^l)$ (Shubin, 1978).

It is necessary to point out that the algebra of invariance of an arbitrary equation $A(y,\partial t,\ldots)f(y)=0$ we name such an algebra of operators $\{Q_\lambda(y,\partial_y,\ldots)\}$ that for any Q_λ $(\lambda=1,\ldots)$ the following holds:

$$[A(y,\partial_y,\ldots), Q_\lambda(y,\partial_y,\ldots)]\bar{f}(y) = 0 \tag{3}$$

where $f(y)$ is an arbitrary solution of (1), and $[A,Q]=AQ-QA$ (Ovsiannikov, 1978; Ibragimav, 1983; Barut, 1977; Fushich, 1978; Man'co, 1979; Rustamov, 1983, 1987).

Now it is easy to prove the following propositions:

1. Equation (1) allows an $(l+1)$-dimensional algebra of invariants $\{\partial_0, P_1,\ldots,P_l\}$.
2. In the case $g^{\mu\mu}=g^{\nu\nu}$ the algebra of invariance allowed by equation (1) also contains the set of transformations of the form $x_\mu P_\nu - x_\nu P_\mu$ $(\nu,\mu=1,\ldots,l)$.
3. If $2n=k$, then the set of transformations of the form $atP_\mu - x_\mu b\partial_0$ (where $a/b=-ig^{\mu\mu}$) is also allowed by equation ((1).

From the point of view of different applications, the following statement is very important.

Theorem. Equation (1) allows the Heisenberg-Weyl algebra of invariance of the operators $\tilde{I}$, $\tilde{P}_\mu$, $\tilde{B}_\mu$ of the following form:

$$\tilde{I}=I; \qquad \tilde{P}_\mu = P_\mu; \qquad \tilde{\beta}_\mu = x_\mu + 2\sigma t(n/k)\mathscr{H}^{(n/k)}(g^{\mu\mu})P_\mu \tag{4}$$

where I is the identity transformation and σ is a constant.

It is easy to prove the theorem by substituting the operators (4) in the conditions (3) for invariance of equation (1).

According to the Stone (1930)-von Neumann (1931) theorem, in the space of dynamical systems under consideration, there are no invariant subspaces of the operators (4). By transformations whose generators are the above-mentioned operators, from only one state of the dynamical system all others can be constructed.

Furthermore, analogously to the problem of the quantum oscillator (Man'co, 1979; Rustamov, 1983; Glauber, 1963), one can obtain coherent states for the system as eigenfunctions of linear combinations of operators (4), and then the corresponding Green's function, etc.

REFERENCES

Barut, A. O. (1977). *Theory of Group Representations and Applications*, PWN, Warsaw.

Fushich, V. I. (1978). In *Sb. Nauch. trudov "Teoretiko-gruppovie metodi v matematischeskoy fisike*, IM AN USSR, Kiev.

Glauber, R. J. (1963). *Physical Review Letters* **10**, 83.

Ibragimav, N. H. (1983). *Gruppi preobrasovaniy v matematicheskoy fisike*, Nauka, Moscow.

Man'co, V. I. (1979). *Dinamicheskie simmetrii i kogerentnie sostoianiya kvantovikh sistem*, Nauka, Moscow.

Ovsiannikov, L. V. (1978). *Gruppovoi analis differenchialnich uravneniy*, Nauka, Moscow.

Rustamov, K. A. (1983). In *Lecture Notes in Physics*, No. 180, *Group Theoretical Methods in Physics*, p. 393, Springer-Verlag, New York.

Rustamov, K. A. (1987). Algebra of invariants, coherent states and Green function for the Kane model, *Journal of Physics C*, submitted.

Shubin, M. A. (1978). *Psevdodifferensialnie operatori i spectralnaia teoriya*, Nauka, Moscow.

Stone, M. (1930). *Proceedings of the National Academy of Sciences of the USA*, **16**, 172.

Von Neumann, J. (1931). *Mathematische Annalen*, **104**, 570.